# Commognitive Analysis of Undergraduate Mathematics Students' Responses in Proving Subgroup's Non-Emptiness

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Proving that a given set is indeed a subgroup, one needs to show that it is non-empty, and closed under operation and inverses. This study focuses on the first condition, analysing students' responses to this task. Results suggest that there are three distinct problematic responses: the total absence of proving this condition, the problematic understanding of subgroup's definition, and the inaccurate application of the relevant metarules. For the purposes of this study there has been used the Commognitive Theoretical Framework.

## Introduction

Research in the learning of Group Theory proves significant, since novice students consider this module as one of the most demanding subjects in their syllabus. It "is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Group Theory module requires a deep understanding of the abstract concepts involved, namely group, subgroup, coset, normal subgroup, quotient group etc. An important element that causes students' difficulty with Group Theory is its abstract nature (Hazzan, 2001). The deductive way of teaching Group Theory is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p. 140). In addition, Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics, which can still occur in their second year. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this module (Ioannou, 2012). The aim of this study is to investigate the undergraduate mathematics students' responses to the concept of subgroup, and in particular in proving non-emptiness, during their first encounter with Group Theory in their second year. For the purposes of this study, there has been used the Commognitive Theoretical Framework (CTF) (Sfard, 2008), due to its great potential to investigate mathematical learning in both object level and meta-discursive level (Presmeg, 2016).

## Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). Moreover, CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated metarules, namely the

2016. In White, B., Chinnappan, M. & Trenholm, S. (Eds.). Opening up mathematics education research (Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia), pp. 344–351. Adelaide: MERGA.

*how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers*, *realisation trees*, *realisations*, *primary objects* and *discursive objects*. It also involves the constructs of *object-level* and *metadiscursive level* (or metalevel) *rules*. Thinking "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p. 81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al., 2014).

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p. 166) The *realization tree* is a "hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth" (Sfard, 2008, p300).

Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel discourse learning*. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254).

CTF has proved particularly appropriate for the purposes of this study, since, as Presmeg (2016, p. 423) suggests, it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate "the entire fabric of human development and what it means to be human."

## Literature Review

Research in the learning of Group Theory is relatively scarce compared to other university mathematics fields, such as Calculus, Linear Algebra or Analysis. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al., 2014). In the context of this research strand, Ioannou (2012) has, among other issues, focused on the intertwined nature of object-level and meta-level learning in Group Theory and the commognitive conflicts that emerge.

The first reports on the learning of Group Theory appeared in the early 1990's. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic concepts.

The construction of the newly introduced d-object of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty with the construction of the Group Theory concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p. 169), since historically the fundamental concepts of Group Theory were permutation and symmetry (Carspecken, 1996). Moreover, this chasm of ontological and historical development proves to be of significant importance in the metalevel development of the group-theoretic discourse for novice students.

From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding of the learning of Group Theory due both to the ontological characteristics of Group Theory, as well as the epistemological tenets of CTF. Group Theory can be considered as a metalevel development of the theory of permutations and symmetries, and CTF allows us to consider the historical and ontological development of a rather "historically decontextualized" modern presentation of this Theory.

Research suggests that students' understanding of the d-object of group proves often primitive at the beginning, predominantly based on their conception of a set. An important step in the development of the understanding of the concept of group is when the student "singles out the binary operation and focuses on its function aspect" (Dubinsky et al., 1994, p292). Students often have the tendency to consider group as a "special set", ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. These last conclusions were based on students' encounter with groups presented in the form of group tables. In fact, students when using group tables adopt various methods for reducing the level of abstraction, by retreating to familiar mathematical structure, by using canonical procedure, and by adopting a local perspective (Hazzan, 2001).

An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Group Theory, and problematic conception of the symbols used probably causes confusion in other instances.

A distinctive characteristic of university mathematics is the production of rigorous and consistent proofs. Proof production is far from a straightforward task to analyse and identify the difficulties students face. These difficulties have been extensively investigated for various levels of student expertise. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically use it.

## Methodology

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Abstract Algebra. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year. This module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Naturally all sources of data have been appropriately analysed, and the conclusions of the data analysis have been triangulated.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

## Data Analysis

The first problematic response in the process of proving non-emptiness that occurred in seven out of thirteen (7/13) students' solutions was its very absence. Not proving that a prospective subgroup is non-empty possibly indicates students' unawareness of its importance. Figure 1 shows an illustrative example, taken from Student A's solution, of an overall successful application of the routine for a set to be a subgroup, yet lacking the proof of non-emptiness.

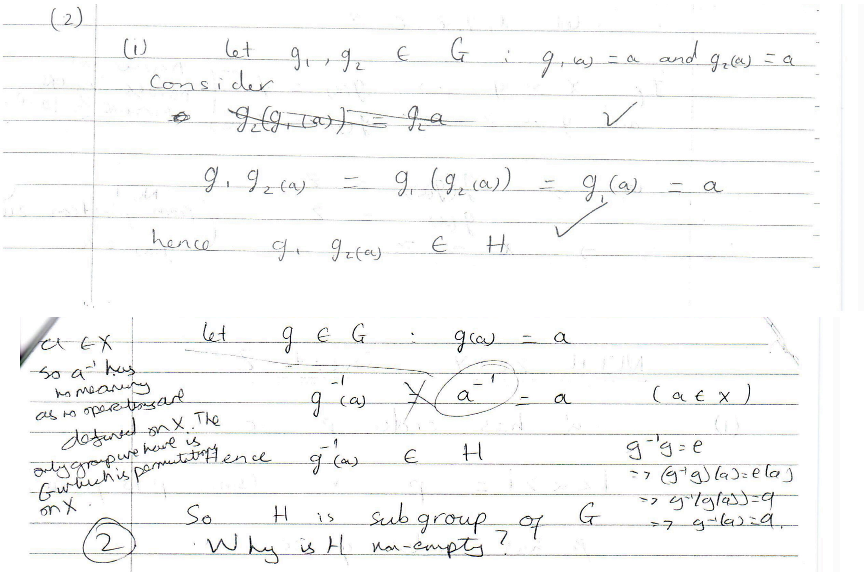

*Figure 1. Solution of Student A*

Regarding the proof of non-emptiness, as the excerpt above demonstrates, even though this student has shown good object-level understanding of the definition of subgroup and capability to generally apply some aspects of the routine for proving that a set is indeed a subgroup, he simply ignored the first condition and considered it valid without any attempt to prove it. The reason for this omission is possibly based on problematic understanding of the governing metarules of the particular routine. Metalevel understanding, namely the understanding of the governing metarules or, in other words, the established "norms" of the process of proving (Sfard, 2008), seems not to be fully adopted by novice students yet. The lack of precision and the omission of the first condition in the particular routine, possibly indicate inadequate evaluation of the importance of proving non-emptiness, or considering that non-emptiness is obvious. The following excerpt is an example from the student interviews, in which Student B reveals the confusion that students probably face, regarding the application of the routine for proving that a set is in fact a subgroup.

> It was just – the problem with that is, getting a – kind of vaguely got it right, but it's just a bit proving you need to really make it crystal-clear, and I'm not too sure if I've done that, made it really clear, what I'm – definitely proving. I mean, say like – the – I'm thinking some of the proofs are probably going through tests for being subgroup, like those, I mean I'll get some marks, but I don't think I'll get the whole – I don't know, when I just write down – well I think I've done the first two parts, don't think were too bad – it's more – say I drew that out, I haven't actually got into testing yet.

Apart from the seven students who did not attempt to prove non-emptiness, three students tried to prove it yet unsuccessfully, with indication of problematic metaphor from Set Theory. In particular, Student C correctly proves closure under operation and closure under inverses showing efficiency in the application of metalevel rules regarding the proof for a set to be a subgroup. Yet, her attempt to prove non-emptiness indicates problematic object-level understanding of the idea of non-emptiness, which resulted inexplicitness in her mathematical narratives and overall reasoning. She proposes that $H \cap K$ is non-empty since if $b \in H$ then $b \in K$ as $H$ and $K$ are both subgroups of $G$ and therefore $b \in G$ so $b \in H \cap K$, which is wrong. She was expected to state that the group identity belongs to the subgroup and therefore it is non-empty. Her answer in Figure 2 indicates problems with the closure of elements of the different subgroups, possibly an erroneous metaphor from Set Theory.

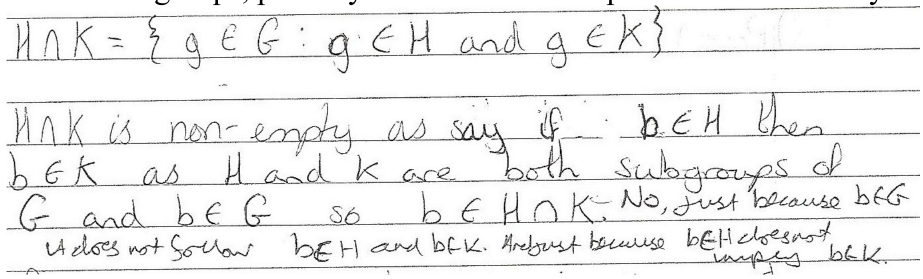


*Figure 2. Solution of Student C*

Similarly to the above, Student D correctly proved closure under operation and closure under inverses. In both examples though, shown in Figures 3 and 4, her proof for non-emptiness is problematic due to object-level understanding of the definition of subgroup. There are several inaccuracies: First of all she considers every $z \in \mathbb{C}^*$ to satisfy the condition $|z| = 1$, without proving that there is such $z$ belonging to the group which satisfies this condition. Another inaccuracy occurs in Figure 4 while trying to prove non-emptiness. The inaccuracy is trivial and is inherited as a problematic metaphor from arithmetic. She concluded that the exponential expression is equal to zero, which is not possible. Both these errors indicate problematic object-level understanding, since these errors are not related to the process of proving and the governing metalevel rules, but with the very d-object of subgroup and the ideas of closure of elements in a set. Student D shows that she knows what steps she should follow to prove non-emptiness yet her understanding of the d-object of subgroup does not allow her to do so.

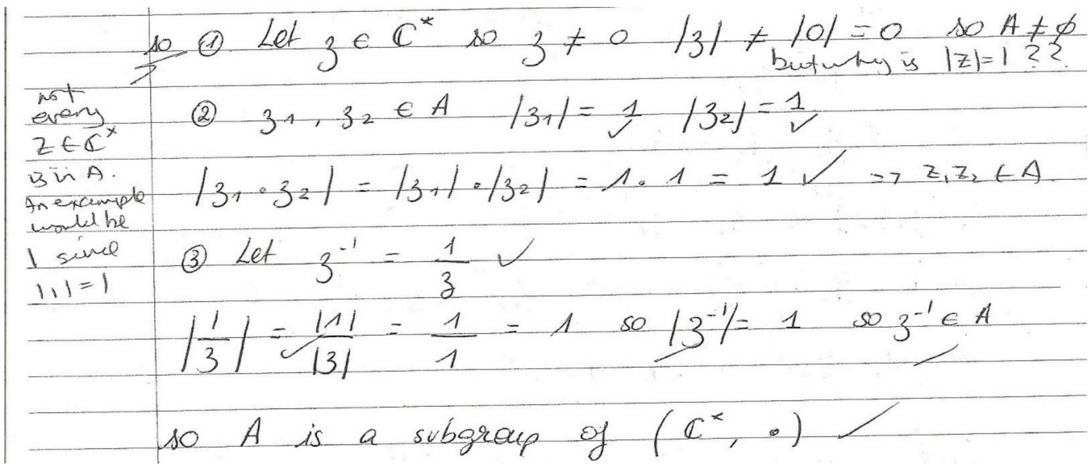


*Figure 3. First example of Student D's solution*

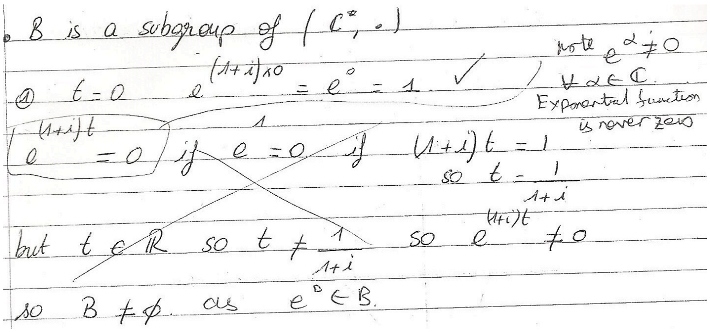


*Figure 4. Second example of Student D's solution*

Although the lack of proving non-emptiness may possibly indicate incomplete object level understanding, the three cases of students who did it unsuccessfully indicate additionally problematic application of the related metarules. In addition to Student C and D, Student E's case is interesting regarding the application of this particular routine, since he is the only student who used visualisation, namely visual metaphors from Complex Analysis and Set Theory, in substitution to a formal "algebraic" answer. Although he is aware of the test for a set to be a subgroup, he attempts to prove non-emptiness by offering, on two occasions, visual mediators (an Argand diagram and Venn diagrams). The marker considers this as inadequate reasoning, as seen in Figure 5. Instead, the student should have given an example of an element.

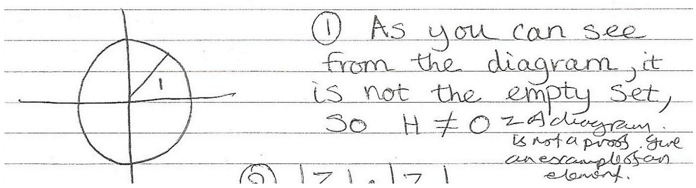


*Figure 5. Solution of Student E*

Interestingly, Student E seems aware that using only visual mediators is not adequate for solving the mathematical problem.

> Err – see, there – this thing – statement, makes sense – I drew a little picture and I was like, I mean – course that's going to be in it, but, how you prove that by actual kind of – prove that mathematically rather than just drawing a picture and just saying, it is true, it's just the actual showing that...

From a more macroscopic viewpoint, looking at all 78 students on the module, a problematic deal with proving non-emptiness is also highlighted by the seminar assistants' comments in their report on the students' coursework results, as can be seen in Figure 6. Their comments were as follows:

- Some lost marks by forgetting to check that the set is non-empty. Giving a concrete example is sufficient e.g $1 \in \{z \in \mathbb{C}^{\times} : |z| = 1\}$.

Q4 - Again most people correctly applied the subgroup criteria. Some lost marks on showing that $H \cap K$ is non-empty. The fact that $H$ and $K$ are non-empty is not sufficient, you need to justify that they share at least one element (namely the identity element).

*Figure 6. Comments of Seminar Assistants*

# Discussion

The proof or non-proof of non-emptiness has revealed for different students, different problems in their object-level and/or metalevel understanding of the d-object of subgroup. Firstly, at this early stage of their encounter with Group Theory students do not realise the importance of certain aspects of the routine for proving that a given set is indeed a subgroup. In particular, proof of non-emptiness was often considered as an established fact, with no obvious need to explicitly prove it. A second type of problematic response in proving non-emptiness was related to students' incomplete understanding of the definition of the d-object of subgroup, partly depending on problematic metaphors from other mathematical discourses such as Set Theory, Complex Analysis and Arithmetic. It seems that well-objectified concepts, from mathematical theories, are considered useful tools and students tend to apply them in new mathematical discourses such as Group Theory. Elements of metaphors that are perceived as relatively concrete and "accessible", such as Venn diagrams and Argand diagrams are more likely to appear as part of the solution. The third type of problematic response is associated with the application of the routine's governing metarules. It is apparent that students in their effort to produce a reasonable proof occasionally use visual mediators in order to substantiate their argumentation. The use of visual mediators as a substitute to an algebraic proof probably indicates incomplete understanding of the required metarules.

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